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Asymptotic Behaviour of Best l_p-Approximations from Affine Subspaces[1]

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In this paper we consider the problem of best approximation in ℓ_p^n, $1<p\leqslant\infty$. If h_p, $1<p<\infty$, denotes the best ℓ_p-approximation of the element $h\in\mathbb{R}^n$ from a proper affine subspace K of $\mathbb{R}^n$, $h\notin K$, then $\lim_{p\to\infty}h_p=h_\infty^*$, where h_∞^* is a best uniform approximation of h from K, the so-called strict uniform approximation. Our aim is to prove that for all $r\in\mathbb{N}$ there are $\alpha_j\in\mathbb{R}^n$, $1\leqslant j\leqslant r$, such that

$$h_p = h_\infty^* + \frac{\alpha_1}{p-1} + \frac{\alpha_2}{(p-1)^2} + \cdots + \frac{\alpha_r}{(p-1)^r} + \gamma_p^{(r)},$$

with $\gamma_p^{(r)}\in\mathbb{R}^n$ and $\|\gamma_p^{(r)}\| = \mathcal{O}(p^{-r-1})$. © 2002 Elsevier Science (USA)

Key Words: strict best approximation; rate of convergence; Polya algorithm; asymptotic expansion.

1. INTRODUCTION

For $x=(x(1),x(2),\ldots,x(n))\in\mathbb{R}^n$, the ℓ_p-norms, $1\leqslant p\leqslant\infty$, are defined by

$$\|x\|_p = \left(\sum_{j=1}^{n} |x(j)|^p\right)^{1/p}, \qquad 1\leqslant p<\infty,$$

$$\|x\| := \|x\|_\infty = \max_{1\leqslant j\leqslant n} |x(j)|.$$

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Let $K \neq \emptyset$ be a subset of $\mathbb{R}^n$. For $h \in \mathbb{R}^n \backslash K$ and $1 \leqslant p \leqslant \infty$ we say that $h_p \in K$ is a best ℓ_p-approximation of h from K if

$$||h_p - h||_p \leqslant ||f - h||_p \quad \text{for all } f \in K.$$

If K is a closed set of $\mathbb{R}^n$, then the existence of h_p is guaranteed. Moreover, there exists a unique best ℓ_p-approximation if K is a closed convex set and $1 < p < \infty$. Throughout this paper, K denotes a proper affine subspace of $\mathbb{R}^n$. Without loss of generality we will assume that $h = 0$ and $0 \notin K$. It is well known (see for instance [8]) that h_p, $1 < p < \infty$, is the best ℓ_p-approximation of 0 from K if and only if

$$\sum_{j=1}^{n} (h_p(j) - f(j))|h_p(j)|^{p-1} \operatorname{sgn}(h_p(j)) = 0 \qquad \text{for all } f \in K. \tag{1}$$

Writing $K = f_0 + \mathscr{V}$ for some $f_0 \in K$ and $\mathscr{V}$ a linear subspace of $\mathbb{R}^n$, then (1) is just equivalent to

$$\sum_{j=1}^{n} v(j)|h_p(j)|^{p-1} \operatorname{sgn}(h_p(j)) = 0 \qquad \text{for all } v \in \mathscr{V}. \tag{2}$$

In the case $p = \infty$ we will say that h_∞ is a best uniform approximation of 0 from K. In general, the unicity of the best uniform approximation is not guaranteed. However, an unique "strict uniform approximation," h^*_∞, can be defined [4, 7]. It is known [1, 5, 7] that if K is an affine subspace of $\mathbb{R}^n$, then

$$\lim_{p \to \infty} h_p = h^*_\infty.$$

In the literature, the convergence above is called Polya algorithm and occurs at a rate no worse than $1/p$, (see [2, 5]). The aim of this paper is to prove that the best ℓ_p-approximation h_p has an asymptotic expansion of the form

$$h_p = h^*_\infty + \frac{\alpha_1}{p-1} + \frac{\alpha_2}{(p-1)^2} + \cdots + \frac{\alpha_r}{(p-1)^r} + \gamma_p^{(r)},$$

for some $\alpha_j \in \mathbb{R}^n$, $1 \leqslant j \leqslant r$, $\gamma_p^{(r)} \in \mathbb{R}^n$ and $||\gamma_p^{(r)}|| = \mathcal{O}(p^{-r-1})$.

In [5] the authors give a necessary and sufficient condition on K for

$$p||h_p - h^*_\infty|| \to 0 \qquad \text{as } p \to \infty \tag{3}$$

and in [6] it is proved that if (3) holds then there is a number $0 < a < 1$ such that $p\,||h_p - h^*_\infty||/a^p$ is bounded. In particular, this result implies that if (3) holds, then we have an exponential rate of convergence of h_p to h^*_∞ as $p \to \infty$ and so the asymptotic expansion of h_p follows immediately with

$\alpha_l = 0$, $1 \leqslant l \leqslant r$, for all $r \in \mathbb{N}$. In the next section, as a consequence of Theorem 2.1, we will deduce the conditions on K such that this situation occurs.

2. NOTATION AND PRELIMINARY RESULTS

Without loss of generality, we will assume that $||h^*_\infty|| = 1$, $h^*_\infty(j) \geqslant 0$, $1 \leqslant j \leqslant n$, and that the coordinates of h^*_∞ are in decreasing order. Let $1 = d_1 > d_2 > \cdots > d_s \geqslant 0$ denote all the different values of $h^*_\infty(j)$, $1 \leqslant j \leqslant n$, and $\{J_l\}_{l=1}^s$ the partition of $J := \{1, 2, \ldots, n\}$ defined by $J_l := \{j \in J : h^*_\infty(j) = d_l\}$, $1 \leqslant l \leqslant s$. We henceforth put $s_0 = s$ if $d_s > 0$ and $s_0 = s - 1$ if $d_s = 0$.

We can write $K = h^*_\infty + \mathscr{V}$, where $\mathscr{V}$ is a proper linear subspace of $\mathbb{R}^n$. It is possible to choose a basis $\mathscr{B} = \{v_1, v_2, \ldots, v_m\}$ of $\mathscr{V}$ and a partition $\{I_k\}_{k=1}^s$ of $I := \{1, 2, \ldots, m\}$ such that for all $i \in I_k$, $1 \leqslant k \leqslant s$,

(p1) $v_i(j) = 0$, $\forall j \in J_l$, $1 \leqslant l < k$,

(p2) $v_i(j) \neq 0$ for some $j \in J_k$.

The set of indices I_k could be empty for some k, $1 \leqslant k \leqslant s$. However, for simplicity of notation, we suppose that $I_k \neq \emptyset$ for $1 \leqslant k \leqslant s_0$, this involves no loss of generality.

We will use the following result [5, 6].

Theorem 2.1. *Under the above conditions, let*

$$a = \max_{1 \leqslant l,k \leqslant r} \left\{ d_l/d_k : \sum_{j \in J_l} v_i(j) \neq 0 \quad \textit{for some } i \in I_k \right\}, \tag{4}$$

where a is assumed to be 0 *if* $\sum_{j \in J_l} v_i(j) = 0$ *for all* $i \in I_k$, $1 \leqslant k, l \leqslant s_0$. *Then there are* $L_1, L_2 > 0$ *such that*

$$L_1 a^p \leqslant p\, ||h_p - h^*_\infty|| \leqslant L_2 a^p, \qquad \forall p > 1. \tag{5}$$

The following notation will be also used in the next section. We put $I_0 = \bigcup_{k=1}^{s_0} I_k$, $m_0 = \mathrm{card}(I_0)$, $J_0 = \bigcup_{l=1}^{s_0} J_l$ and we consider the matrices $M = (v_i(j))_{(i,j) \in I_0 \times J_0}$ and $M_{kl} = (v_i(j))_{(i,j) \in I_k \times J_l}$, $1 \leqslant k, l \leqslant s_0$. Finally, we denote by A^T the transpose of the matrix A and by $||A||$ the row-sum norm of A.

Lemma 2.1. *If* $\{x_p\}$ *is a sequence of real numbers such that* $p\,|x_p| \to 0$ *as* $p \to \infty$, *then*

$$(1 + x_p)^p = 1 + p\, x_p + R_p,$$

with $R_p = o(p\,|x_p|)$.

Proof. The proof follows immediately from the application of Taylor's formula to the function $\varphi(z) = (1+z)^p$ at $z = 0$. ∎

In the next formula we use the following standard notation. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$. If $\mathbf{r} = (r_1, r_2, \ldots, r_k) \in \mathbb{N}_0^k$ and $\mathbf{a} = \{a_j\}_{j\in\mathbb{N}}$ is a sequence of real numbers, then we define $|\mathbf{r}| := r_1 + r_2 + \cdots + r_k$, $\mathbf{r}! := r_1!r_2!\cdots r_k!$ and $\mathbf{a}^{\mathbf{r}} = a_1^{r_1} a_2^{r_2} \cdots a_k^{r_k}$. Also, for $i \in \mathbb{N}$, we denote $\mathscr{G}(k,i) := \{\mathbf{r} \in \mathbb{N}_0^k : \sum_{j=1}^k jr_j = i\}$.

Let $\mathbf{a} = \{a_j\}_{j\in\mathbb{N}}$ and $\mathbf{b} = \{b_j\}_{j\in\mathbb{N}}$ be two sequences of real numbers and $m, n \in \mathbb{N}$. An easy computation gives

$$f_{m,n}(z) := \sum_{j=1}^{n} b_j \left(\sum_{i=1}^{m} a_i z^i \right)^j = \sum_{i=1}^{mn} \sum_{r\in\mathscr{G}(m,i)} \frac{|\mathbf{r}|!}{\mathbf{r}!} b_{|\mathbf{r}|}\, \mathbf{a}^{\mathbf{r}}\, z^i.$$

Applying the above formula and the Rolle Theorem we easily obtain the expansion of known functions. For example, taking $b_j = 1/j!$, $j = 1, 2, \ldots,$ we get

$$\exp\,[a_1 z + \cdots + a_k z^k] = 1 + \sum_{i=1}^{k} \sum_{r\in\mathscr{G}(k,i)} \frac{\mathbf{a}^{\mathbf{r}}}{\mathbf{r}!} z^i + \mathcal{O}(z^{k+1}).$$

Analogously, taking $b_j = (-1)^{j-1}/j$, $j = 1, 2, \ldots,$ we have

$$\frac{1}{z}\log(1 + a_1 z + \cdots + a_k z^k) = \sum_{i=1}^{k+1} \sum_{r\in\mathscr{G}(k,i)} \frac{(-1)^{|\mathbf{r}|-1}(|\mathbf{r}|-1)!}{\mathbf{r}!} \mathbf{a}^{\mathbf{r}} z^{i-1} + \mathcal{O}(z^{k+1}).$$

Now we could use the formulas above to obtain explicitly the asymptotic expansion of order k of the expression

$$\left(1 + \frac{a_1}{p} + \cdots + \frac{a_k}{p^k}\right)^p = \exp\left[p \log\left(1 + \frac{a_1}{p} + \cdots + \frac{a_k}{p^k}\right)\right].$$

However, in order to simplify the notation, we resume these observations in the following result.

LEMMA 2.2. *Let $k \in \mathbb{N}$ and $a_l \in \mathbb{R}$, $1 \leqslant l \leqslant k$. Then there are $c_l \in \mathbb{R}$, $1 \leqslant l \leqslant k$, with $c_l = c_l(a_1, \ldots, a_l)$, such that*

$$\left(1 + \frac{a_1}{p} + \cdots + \frac{a_k}{p^r}\right)^p = c_0 + \frac{c_1}{p} + \cdots + \frac{c_k}{p^k} + \mathcal{O}\left(\frac{1}{p^{k+1}}\right),$$

where $c_0 = e^{a_1}$.

For $v \in \mathscr{V}$, $v \neq 0$, and $1 \leqslant t \leqslant s$, let $J_t[v]$ be the set of indices j in J_t such that $v(j) \neq 0$ and define

$$t_v := \min\{t \in \{1, \ldots, s\} : J_t[v] \neq \emptyset\} \quad \text{and} \quad \hat{J}_{t_v} := J_{t_v}[v].$$

Also, if $J' \subset J$ we denote by $||\cdot||_{J'}$ the restriction of the norm $||\cdot||$ to the set of indices in J'.

LEMMA 2.3. *Suppose that there are $\alpha_l \in \mathscr{V}$, $1 \leqslant l \leqslant r$, such that*

$$h_p = h_\infty^* + \sum_{l=1}^{r} \frac{\alpha_l}{(p-1)^l} + \gamma_p^{(r)},$$

where $(p-1)^\tau \gamma_p^{(r)} \to 0$ as $p \to \infty$ for some $\tau \in \mathbb{N}$. Let $v \in \mathscr{V}$, $v \neq 0$, and suppose that $||\alpha_l||_{\hat{J}_{t_v}} \neq 0$ for some $l \in \{0, 1, \ldots, r\}$, where $\alpha_0 := h_\infty^*$. Define

$$l_v := \min\{l \in \{0, 1, \ldots, r\} : \alpha_l(j) \neq 0 \quad \text{for some } j \in \hat{J}_{t_v}\}$$

and let $\hat{J}^0_{t_v}$ be the set of indices in $\hat{J}_{t_v}$ such that $|\alpha_{l_v}(j)| = ||\alpha_{l_v}||_{\hat{J}_{t_v}}$. Then

$$\sum_{j \in \hat{J}^0_{t_v}} v(j) c_l(j) \operatorname{sgn}(\alpha_{l_v}(j)) = 0, \qquad 0 \leqslant l \leqslant \tau - l_v - 1, \tag{6}$$

where, for each $j \in \hat{J}_{t_v}$, the coefficients $c_l(j)$ are given by Lemma 2.2 with $a_l = \alpha_{l+l_v}(j)/\alpha_{l_v}(j)$, $1 \leqslant l \leqslant r - l_v$ and $k = r - l_v$.

Proof. Note that we can assume that $l_v < \tau$; otherwise the condition in (6) is empty. Applying (2), we have

$$\sum_{j \in J} v(j) |h_p(j)|^{p-1} \operatorname{sgn}(h_p(j)) = 0,$$

and so, multiplying by $((p-1)^{l_v} / ||\alpha_{l_v}||_{\hat{J}_{t_v}})^{p-1}$,

$$\sum_{j \in J} v(j) \left| (p-1)^{l_v} \frac{h_p(j)}{||\alpha_{l_v}||_{\hat{J}_{t_v}}} \right|^{p-1} \operatorname{sgn}(h_p(j)) = 0. \tag{7}$$

If $j \in \hat{J}^0_{t_v}$, then, since $(p-1)^{l_v} \gamma_p^{(r)}(j) \to 0$ as $p \to \infty$, we have, for p large enough, $\operatorname{sgn}(h_p(j)) = \operatorname{sgn}(\alpha_{l_v}(j))$ and

$$(p-1)^{l_v} \frac{|h_p(j)|}{||\alpha_{l_v}||_{\hat{J}_{t_v}}} = 1 + \sum_{l=l_v+1}^{r} \frac{\alpha_l(j)/\alpha_{l_v}(j)}{(p-1)^{l-l_v}} + (p-1)^{l_v} \frac{\gamma_p^{(r)}(j)}{\alpha_{l_v}(j)}.$$

Also, since $(p-1)^{l_v+1}\gamma_p^{(r)}(j) \to 0$ as $p \to \infty$, we can apply Lemma 2.1 to obtain

$$\begin{aligned}
&\left(1+\sum_{l=l_v+1}^{r}\frac{\alpha_l(j)/\alpha_{l_v}(j)}{(p-1)^{l-l_v}}+(p-1)^{l_v}\frac{\gamma_p^{(r)}(j)}{\alpha_{l_v}(j)}\right)^{p-1}\\
&\quad=\beta_p(j)^{p-1}\left(1+(p-1)^{l_v}\frac{\gamma_p^{(r)}(j)}{\beta_p(j)\alpha_{l_v}(j)}\right)^{p-1}\\
&\quad=\beta_p(j)^{p-1}+(p-1)^{l_v+1}\beta_p(j)^{p-2}\frac{\gamma_p^{(r)}(j)}{\alpha_{l_v}(j)}(1+R_p(j)),
\end{aligned}\tag{8}$$

where $R_p(j)=o(1)$ and $\beta_p(j)=1+\sum_{l=l_v+1}^{r}\frac{\alpha_l(j)/\alpha_{l_v}(j)}{(p-1)^{l-l_v}}$.

Now, from Lemma 2.2, we have

$$\beta_p(j)^{p-1}=\left(1+\sum_{l=l_v+1}^{r}\frac{\alpha_l(j)/\alpha_{l_v}(j)}{(p-1)^{l-l_v}}\right)^{p-1}=\sum_{l=0}^{r-l_v}\frac{c_l(j)}{(p-1)^l}+E_p(j),\tag{9}$$

with $E_p(j)=\mathcal{O}((p-1)^{l_v-r-1})$ and the coefficients $c_l(j)$, $0\leqslant l\leqslant r-l_v$, depend on $\alpha_l(j)$, $l_v\leqslant l\leqslant r$.

On the other hand, if $j\in J\backslash\hat{J}^0_{t_v}$ and $v(j)\neq 0$, then

$$\begin{aligned}
&\lim_{p\to\infty}(p-1)^{l_v}\frac{|h_p(j)|}{||\alpha_{l_v}||_{\hat{J}_{t_v}}}\\
&\quad=\lim_{p\to\infty}\left|\sum_{l=l_v}^{r}\frac{\alpha_l(j)/||\alpha_{l_v}||_{\hat{J}_{t_v}}}{(p-1)^{l-l_v}}+(p-1)^{l_v}\frac{\gamma_p^{(r)}(j)}{||\alpha_{l_v}||_{\hat{J}_{t_v}}}\right|=\frac{|\alpha_{l_v}(j)|}{||\alpha_{l_v}||_{\hat{J}_{t_v}}}<1.
\end{aligned}$$

From (8) and (9), Eq. (7) can be written as

$$\begin{aligned}
&\sum_{j\in J\backslash\hat{J}^0_{t_v}}v(j)\left|(p-1)^{l_v}\frac{h_p(j)}{||\alpha_{l_v}||_{\hat{J}_{t_v}}}\right|^{p-1}\operatorname{sgn}(h_p(j))\\
&\quad+\sum_{j\in\hat{J}^0_{t_v}}\sum_{l=0}^{r-l_v}\frac{v(j)c_l(j)}{(p-1)^l}\operatorname{sgn}(\alpha_{l_v}(j))+\sum_{j\in\hat{J}^0_{t_v}}u(j)E_p(j)\operatorname{sgn}(\alpha_{l_v}(j))\\
&\quad+(p-1)^{l_v+1}\sum_{j\in\hat{J}^0_{t_v}}v(j)\beta_p(j)^{p-2}\frac{\gamma_p^{(r)}(j)}{|\alpha_{l_v}(j)|}(1+R_p(j))=0.
\end{aligned}\tag{10}$$

Finally, multiplying (10) by $(p-1)^l$, $0\leqslant l\leqslant \tau-l_v-1$, and taking limits as $p\to\infty$ we conclude (6). ∎

If $l_v \leqslant \tau$, taking into account (6) and multiplying (10) by $(p-1)^{\tau-l_v}$, we can write for short,

$$\sum_{j\in\hat{J}^0_{t_v}} v(j)c_{\tau-l_v}(j)\,\mathrm{sgn}(\alpha_{l_v}(j)) + (p-1)^{\tau+1}\sum_{j\in\hat{J}^0_{t_v}} v(j)\beta_p(j)^{p-2}\frac{\gamma_p^{(r)}(j)}{|\alpha_{l_v}(j)|}(1+R_p(j)) + W_p = 0, \quad (11)$$

where $W_p = o(1)$.

3. ASYMPTOTIC BEHAVIOUR OF BEST ℓ_p-APPROXIMATIONS

THEOREM 3.1. *Let K be a proper affine subspace of* $\mathbb{R}^n$, $0 \notin \mathbb{K}$. *For* $1<p<\infty$, *let* h_p *denote the best* ℓ_p*-approximation of* 0 *from K and let* h^*_∞ *be the strict uniform approximation. Then, for all* $r \in \mathbb{N}$, *there are* $\alpha_l \in \mathbb{R}^n$, $1\leqslant l\leqslant r$, *such that*

$$h_p = h^*_\infty + \frac{\alpha_1}{p-1} + \cdots + \frac{\alpha_r}{(p-1)^r} + \gamma_p^{(r)}, \quad (12)$$

where $\gamma_p^{(r)} \in \mathbb{R}^n$ *and* $||\gamma_p^{(r)}|| = \mathcal{O}(p^{-r-1})$.

Proof. Since $p\,||h_p - h^*_\infty||$ is bounded [2, 5], the proof follows immediately by induction on r with the help of Lemmas 3.1 and 3.2. ∎

LEMMA 3.1. *Under the same conditions of Theorem* 3.1, *let* $r \in \mathbb{N}$ *and suppose that there are* $\alpha_l \in \mathcal{V}$, $1\leqslant l\leqslant r-1$, *such that*

$$h_p = h^*_\infty + \sum_{l=1}^{r-1}\frac{\alpha_l}{(p-1)^l} + \gamma_p^{(r-1)}.$$

If there exists $\alpha_r := \lim_{p\to\infty}(p-1)^r\gamma_p^{(r-1)}$, *then* $(p-1)^{r+1}||\gamma_p^{(r)}||$ *is bounded, where* $\gamma_p^{(r)} := \gamma_p^{(r-1)} - \alpha_r/(p-1)^r$.

Proof. Obviously, we only need to consider the case $||\gamma_p^{(r)}|| \neq 0$ for p large enough. Let $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ and I_k, $1\leqslant k\leqslant s$, defined as above. By the definition of α_r, we can write

$$h_p = h^*_\infty + \sum_{l=1}^{r}\frac{\alpha_l}{(p-1)^l} + \gamma_p^{(r)}.$$

The definition of α_r also implies, $p^r\,||\gamma_p^{(r)}|| \to 0$ as $p \to \infty$. Then, it is possible to apply Lemma 2.3 with $\tau = r$ and $v = v_i$, $i \in I_k$, $1 \leqslant k \leqslant s_0$. In this case, by (p1) and (p2), $t_{v_i} = k$, $l_{vi} = 0$ and hence, for $j \in \hat{J}_{t_{vi}} = \hat{J}_k$, $\alpha_{l_{vi}}(j) = \alpha_0(j) = h^*_\infty(j) = d_k$. Then, from (11), we have

$$\sum_{j\in J_k} v_i(j)c_r(j) + (p-1)^{r+1} \sum_{j\in J_k} v_i(j)\beta_p(j)^{p-2}\,\frac{\gamma_p^{(r)}(j)}{d_k}(1+R_p(j)) + W_p(i) = 0,$$

where now $\beta_p(j) = 1 + \sum_{l=1}^{r} \frac{\alpha_l(j)/d_k}{(p-1)^l}$.

Note that we have replaced $\hat{J}_k$ by J_k because $v_i(j) = 0$ for $j \in J_k \backslash \hat{J}_k$. For simplicity of notation, the equation above can be rewritten as

$$(p-1)^{r+1} \sum_{j\in J_k} v_i(j)\beta_p(j)^{p-2}\gamma_p^{(r)}(j) = \tilde{B}(i) + \tilde{R}_p(i) + \tilde{W}_p(i), \tag{13}$$

where $\tilde{B}(i) = -d_k \sum_{j\in J_k} v_i(j)c_r(j)$, $\tilde{W}_p(i) = -d_k\, W_p(i)$ and

$$\tilde{R}_p(i) = -(p-1)^{r+1} \sum_{j\in J_k} v_i(j)\beta_p(j)^{p-2}\gamma_p^{(r)}(j)R_p(j) = o((p-1)^{r+1}||\gamma_p^{(r)}||).$$

Next, we transform Eq. (13), for $i \in I_k$, $1 \leqslant k \leqslant s_0$, to a nonsingular linear system of order $m_0 \times m_0$. Indeed, since $\gamma_p^{(r)} \in \mathscr{V}$, there are real numbers $\lambda_p(t)$, $1 \leqslant t \leqslant m$, such that

$$\gamma_p^{(r)} = \sum_{t=1}^{m} \lambda_p(t)v_t.$$

Then from (13), we obtain, for $i \in I_k$, $1 \leqslant k \leqslant s_0$,

$$(p-1)^{r+1} \sum_{j\in J_k} v_i(j)\beta_p(j)^{p-2} \sum_{t=1}^{m} \lambda_p(t)v_t(j) = \tilde{B}(i) + \tilde{R}_p(i) + \tilde{W}_p(i). \tag{14}$$

Observe that the sum on t in (14) extends only for indices $t \in J_l$ with $1 \leqslant l \leqslant k$, because if $j \in J_k$, then $v_t(j) = 0$ for $t \in J_l$ with $l > k$. The set of equations in (14) is a linear system which can be expressed as

$$(p-1)^{r+1} D\Delta_p M^T \Lambda_p^T = \tilde{B} + \tilde{R}_p + \tilde{W}_p, \tag{15}$$

where D is the diagonal matrix by blocks given by

$$D = \begin{pmatrix} M_{11} & 0 & \cdots & 0 \\ 0 & M_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{s_0 s_0} \end{pmatrix},$$

$\Delta_p := (\delta_{ij})_{(i,j)\in J_0\times J_0}$ is the diagonal matrix with $\delta_{jj} = \beta_p(j)^{p-2}$ and $\Lambda_p = (\lambda_p(1), \ldots, \lambda_p(m_0))$. If we denote $A(p) := D\Delta_p M^T$, then an easy computation shows that

$$A := \lim_{p\to\infty} A(p) = \begin{pmatrix} \hat{M}_{11}\hat{M}_{11}^T & 0 & \cdots & 0 \\ \hat{M}_{22}\hat{M}_{12}^T & \hat{M}_{22}\hat{M}_{22}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \hat{M}_{s_0 s_0}\hat{M}_{1s_0}^T & \hat{M}_{s_0 s_0}\hat{M}_{2s_0}^T & \cdots & \hat{M}_{s_0 s_0}\hat{M}_{s_0 s_0}^T \end{pmatrix},$$

where $\hat{M}_{ij}$ is the matrix obtained multiplying each column of M_{ij} by $e^{\alpha_1(j)/(2d_k)}$ if $j \in J_k$. Then

$$\det(A) = \prod_{i=1}^{s_0} \det(\hat{M}_{ii}\hat{M}_{ii}^T) \neq 0$$

and so the matrix $A(p)$ is nonsingular for p large enough. Solving system (15), we get

$$(p-1)^{r+1}\Lambda_p^T = A(p)^{-1}(\tilde{B} + \tilde{R}_p + \tilde{W}_p),$$

and so $(p-1)^{r+1}||\Lambda_p|| \leqslant ||A(p)^{-1}||(||\tilde{B}|| + ||\tilde{R}_p|| + ||\tilde{W}_p||))$. Hence,

$$(p-1)^{r+1}||\Lambda_p||\left(1 - \frac{||A(p)^{-1}||\,||\tilde{R}_p||}{(p-1)^{r+1}||\Lambda_p||}\right) \leqslant ||A(p)^{-1}||(||\tilde{B}|| + ||\tilde{W}_p||).$$

Taking limits as $p \to \infty$ we have

$$\lim_{p\to\infty} (p-1)^{r+1}||\Lambda_p|| \leqslant ||A^{-1}||\,||\tilde{B}||.$$

Similarly,

$$||\tilde{B}|| \leqslant (p-1)^{r+1}||A(p)||\,||\Lambda_p||\left(1+\frac{||\tilde{R}_p||}{(p-1)^{r+1}||A(p)||\,||\Lambda_p||}\right)+||\tilde{W}_p||$$

and therefore

$$\lim_{p\to\infty}\ (p-1)^{r+1}||\Lambda_p|| \geqslant \frac{||\tilde{B}||}{||A||}.$$

Finally, we conclude that

$$\frac{||\tilde{B}||}{||A||} \leqslant \lim_{p\to\infty}\ (p-1)^{r+1}||\Lambda_p|| \leqslant ||\tilde{B}||\,||A^{-1}||.$$

Observe that we have actually proved that $(p-1)^{r+1}|\gamma_p^{(r)}(j)|$ is bounded for $j \in J_0$. Now our proposal will be to prove that $(p-1)^{r+1}|\gamma_p^{(r)}(j)|$ is also bounded, for all $j \in J_s$ (in case that $s \neq s_0$ and $J_s \neq \emptyset$). Suppose the contrary. Using a subsequence if necessary, we consider the vector $u \in \mathbb{R}^n$ whose coordinates are given by

$$u(j) = \lim_{k\to\infty} \frac{\gamma_{p_k}^{(r)}(j)}{||\gamma_{p_k}^{(r)}||}, \qquad 1 \leqslant j \leqslant n.$$

Note that $u \in \mathscr{V}$, $||u|| = 1$ and $u(j) = 0$ if $(p-1)^{r+1}|\gamma_p^{(r)}(j)|$ is bounded. In particular $u(j) = 0$ for all $j \in J_0$ and so $t_u = s$. Applying (2) with $v = u$ and particularizing for $p = p_k$, we get

$$\sum_{j\in\hat{J}_s} u(j)|h_{p_k}(j)|^{p_k-1}\,\mathrm{sgn}(h_{p_k}(j)) = 0, \tag{16}$$

with $h_{p_k}(j) = \sum_{l=1}^{r} \frac{\alpha_l(j)}{(p_k-1)^l} + \gamma_{p_k}^{(r)}(j)$.

Also, observe that $||\alpha_l||_{\hat{J}_s} \neq 0$ for some $l \in \{1, \ldots, r\}$. Otherwise, from (16), we have, for k large enough,

$$\sum_{j\in\hat{J}_s} |u(j)||\gamma_{p_k}^{(r)}(j)|^{p_k-1} = 0,$$

and we obtain a contradiction.

Since $(p-1)^r\|\gamma_p^{(r)}\| \to 0$ as $p \to \infty$, we can apply Lemma 2.3 with $v = u$. In this case, (11) yields,

$$\sum_{j\in \hat{J}_s^0} u(j)c_{r-l_u}(j)\,\mathrm{sgn}(\alpha_{l_u}(j))$$

$$+\ (p-1)^{r+1}\sum_{j\in \hat{J}_s^0} u(j)\beta_p(j)^{p-2}\frac{\gamma_p^{(r)}(j)}{|\alpha_{l_u}(j)|}(1+R_p(j))+W_p=0.$$

Particularizing the equation above for $p = p_k$ and taking limits as $k \to \infty$, we get another contradiction. ∎

LEMMA 3.2. *Under the same conditions of Theorem* 3.1, *let* $r \in \mathbb{N}$ *and suppose that there are* $\alpha_l \in \mathcal{V}$, $1 \leqslant l \leqslant r-1$, *such that*

$$h_p = h_\infty^* + \sum_{l=1}^{r-1}\frac{\alpha_l}{(p-1)^l}+\gamma_p^{(r-1)}.$$

If $(p-1)^r\|\gamma_p^{(r-1)}\|$ *is bounded, then there exists* $\lim_{p\to\infty}(p-1)^r\gamma_p^{(r-1)} \in \mathcal{V}$.

Proof. Since $(p-1)^r\|\gamma_p^{(r-1)}\|$ is bounded, we can take a subsequence $p_k \to \infty$ such that $(p_k-1)^r\gamma_{p_k}^{(r-1)}$ converges. We define

$$\alpha_r := \lim_{k\to\infty}(p_k-1)^r\gamma_{p_k}^{(r-1)},$$

and we set

$$h_p = h_\infty^* + \sum_{l=1}^{r}\frac{\alpha_l}{(p-1)^l}+\gamma_p^{(r)}, \tag{17}$$

where $\gamma_p^{(r)} := \gamma_p^{(r-1)} - \alpha_r/(p-1)^r$.

First, note that $(p-1)^r\|\gamma_p^{(r)}\|$ is also bounded. Now, our claim is that $(p-1)^r\gamma_p^{(r)} \to 0$ as $p \to \infty$. On the contrary, suppose that there exists a subsequence $p'_k \to +\infty$ such that $(p'_k-1)^r\gamma_{p'_k}^{(r)} \to u \neq 0$. We will show that in this case we get a contradiction. Indeed, since $u \in \mathcal{V}$, applying (2) with $v = u$, we have,

$$\sum_{j\in J} u(j)|h_p(j)|^{p-1}\,\mathrm{sgn}(h_p(j)) = 0 \qquad \text{for all } p > 1. \tag{18}$$

Observe that if $t_u = s$ then $\|\alpha_l\|_{\hat{J}_s} \neq 0$, for some $l \in \{1,\ldots,r\}$. Otherwise, particularizing (17) and (18) for $p = p'_k$, we have, for k large enough

$$\sum_{j\in \hat{J}_s} u(j)|h_{p'_k}(j)|^{p'_k-1} = 0$$

and we get a contradiction.

We consider three exhaustive cases.

(a) If $r > 1$ and $l_u \leqslant r - 1$, then we apply Lemma 2.3 with $\tau = r - 1$ and $v = u$. In this case, from (11)

$$\sum_{j \in \hat{J}^0_{t_u}} u(j) c_{r-1-l_v}(j) \operatorname{sgn}(\alpha_{l_u}(j)) + (p-1)^r \sum_{j \in \hat{J}^0_{t_u}} u(j) \beta_p(j)^{p-2} \frac{\gamma_p^{(r)}(j)}{|\alpha_{l_u}(j)|}(1 + R_p(j)) + W_p = 0. \qquad (19)$$

Particularizing (19) for $p = p_k$ and taking limits as $k \to \infty$, we have

$$\sum_{j \in \hat{J}^0_{t_u}} u(j) c_{r-1-l_u}(j) \operatorname{sgn}(\alpha_{l_u}(j)) = 0.$$

Now, taking limits in (19) as $k \to \infty$ with $p = p'_k$, we get

$$\sum_{j \in \hat{J}^0_{t_u}} \frac{u(j)^2}{|\alpha_{l_u}(j)|} \exp\left(\frac{\alpha_{l_u+1}(j)}{\alpha_{l_u}(j)}\right) = 0,$$

which is a contradiction.

(b) If $l_u = r$, then in particular $t_u = s$ and (18) gives

$$\sum_{j \in J \setminus \hat{J}^0_s} u(j) \left| \frac{\alpha_r(j)}{||\alpha_r||_{\hat{J}_s}} + (p-1)^r \frac{\gamma_p^{(r)}(j)}{||\alpha_r||_{\hat{J}_s}} \right|^{p-1} \operatorname{sgn}(\alpha_r(j) + (p-1)^r \gamma_p^{(r)}(j)) + \sum_{j \in \hat{J}^0_s} u(j) \left| 1 + (p-1)^r \frac{\gamma_p^{(r)}(j)}{\alpha_r(j)} \right|^{p-1} \operatorname{sgn}(\alpha_r(j) + (p-1)^r \gamma_p^{(r)}(j)) = 0. \qquad (20)$$

From Lemma 3.1, we deduce that $(p_k - 1)^{r+1} ||\gamma_{p_k}^{(r)}||$ is bounded. Hence particularizing (20) for $p = p_k$, using a subsequence if necessary and taking limits as $k \to \infty$, we obtain

$$\sum_{j \in \hat{J}^0_s} u(j) e^{\omega(j)} \operatorname{sgn}(\alpha_r(j)) = 0,$$

where $\omega(j) = \lim_{k \to \infty} (p_k - 1)^{r+1} \gamma_{p_k}^{(r)}(j)$. In particular we deduce that there exists $j_0 \in \hat{J}^0_s$ such that $u(j_0)\alpha_r(j_0) > 0$. But, if $j \in \hat{J}^0_s$ and $u(j)\alpha_r(j) > 0$, then

$$\lim_{k \to \infty} \left(1 + (p'_k - 1)^r \frac{\gamma_{p'_k}^{(r)}(j)}{\alpha_r(j)} \right) = 1 + \frac{u(j)}{\alpha_r(j)} > 1$$

and $\operatorname{sgn}(\alpha_r(j) + (p'_k - 1)^r \gamma_{p'_k}^{(r)}(j)) = \operatorname{sgn}(u(j))$, for k large enough. On the other hand, if $j \in \hat{J}_s^0$ with $u(j)\alpha_r(j) < 0$ and $|u(j)| > |\alpha_r(j)|$ then for k large enough,

$$\operatorname{sgn}(\alpha_r(j) + (p'_k - 1)^r \gamma_{p'_k}^{(r)}(j)) = \operatorname{sgn}(u(j)).$$

Finally, if $j \in \hat{J}_s^0$ with $u(j)\alpha_r(j) < 0$ and $|u(j)| \leqslant |\alpha_r(j)|$ then

$$\lim_{k\to\infty} \left| 1 + (p'_k - 1)^r \frac{\gamma_{p'_k}^{(r)}(j)}{\alpha_r(j)} \right| = \left| 1 + \frac{u(j)}{\alpha_r(j)} \right| < 1.$$

So, taking limits in (20) as $k \to \infty$, with $p = p'_k$, we obtain a contradiction.

(c) If $r = 1$ and $l_u = 0$, then, in particular $t_u = k$ with $1 \leqslant k \leqslant s_0$. In this case, from (18), we have

$$\sum_{j\in J_k} u(j) \left(1 + \frac{\alpha_1(j)/d_k}{p-1} + \frac{\gamma_p^{(1)}(j)}{d_k} \right)^{p-1} + \sum_{j\in J\setminus J_k} u(j) \left| \frac{h_p(j)}{d_k} \right|^{p-1} \operatorname{sgn}(h_p(j)) = 0.$$

Particularizing this equation for $p = p_k$ and p'_k and taking limits as $k \to \infty$ we get immediately a contradiction. Indeed, for $p = p_k$, we obtain

$$\sum_{j\in J_k} u(j) e^{\alpha_1(j)/d_k} = 0$$

and for $p = p'_k$,

$$\sum_{j\in J_k} u(j) e^{\alpha_1(j)/d_k} e^{u(j)/d_k} = 0.$$

We conclude our assertion, taking into account that $z(e^z - 1) > 0$, for all $z \in \mathbb{R}\setminus\{0\}$.

Remark 3.1. Using the geometric series expansion, one gets for $p > 1$:

$$\frac{1}{p-1} = \frac{1}{p(1-1/p)} = \frac{1}{p} \sum_{i=0} \frac{1}{p^i} = \sum_{i=1} \frac{1}{p^i}.$$

So, after rearranging terms, the expansion in (12) can also be written as one in the standard form

$$h_p = h^*_\infty + \sum_{v=1} \frac{\beta_v}{p^v} + \hat{\gamma}_p^{(r)},$$

where $\|\hat{\gamma}_p^{(r)}\| = \mathcal{O}(p^{-r-1})$.

Remark 3.2. Let us observe that if $\sum_{j\in J_k} v_i(j) = 0$ for all $i \in I_k$ and all $1 \leqslant k \leqslant s_0$, then, from (4), $0 \leqslant a < 1$. In this case, as a consequence of Theorem 2.1, $p^k \|h_p - h^*_\infty\| \to 0$ for all $k \in \mathbb{N}$ and hence (12) holds immediately for $\alpha_l = 0 \in \mathbb{R}^n$, for all $l = 1, \ldots, r$. Therefore, in order to get nontrivial expansions of h_p, we must assume that $\sum_{j\in J_k} v_i(j) \neq 0$ for some $i \in I_k$, $1 \leqslant k \leqslant s_0$.

Remark 3.3. In [3], the authors suggest the asymptotic expansion,

$$h_p = h^*_\infty + \sum_{i=1}^{\infty} \frac{B_i}{p^i}.$$

They apply the series above to obtain good estimations of h^*_∞ by means of extrapolation techniques. However, to our knowledge, there was not any proof of this formula.

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